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ADDENDUM

Relativistic kinematics of interferometry

G C Scorgie
8 Balmoral Place, Gourock, Renfrewshire PA19 1AN, UK

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Abstract. In a stationary metric relations are established between the classical motion of a representative particle and the eikonal for the associated de Broglie waves for particle beams. It is shown that the group velocity of the waves is the same as the coordinate velocity of the particle. An explicit expression for the eikonal in terms of the particle motion resolves a paradox that can result from taking the proper time of the particle to be a measure of the eikonal.

The paper by Scorgie (1993) treated quantum mechanical interference of beams of material particles in terms of the motion of a representative particle assumed to travel on the exactly defined classical trajectory. Wave aspects entered only through the introduction of the angular frequency of the associated de Broglie wave, the phase difference that accounts for interference being calculated as the product of that angular frequency and the difference between the coordinate time of transit of particles on their classical trajectories.

It may be objected that this is a piecemeal procedure that invites closer examination by relating it to the more usual approach in which attention focuses on the wave aspects. This addendum describes such an examination. In particular for a free particle it is shown that the group velocity of the de Broglie waves is equal to the particle coordinate velocity that formed the basis of the treatment in the paper. Of course it would be disturbing if this were not so, but since both velocities are expressed using coordinate measures of distance and time in a non-inertial coordinate system, the equality can scarcely be taken for granted. An explicit expression is found for the phase of the wave in terms of the particle motion and is shown to resolve a paradox that was noted in the paper. The earlier notation is broadly retained and extended as necessary.

The square of the element of interval

$$ds^2 = g_{mn}\, dx^m\, dx^n + 2g_{m4}\, dx^m\, dx^4 + g_{44}(dx^4)^2 \tag{1}$$

can be written

$$ds^2 = d\sigma^2 + 2g_{m4}\lambda^m\, d\sigma\, dx^4 + g_{44}(dx^4)^2 \tag{2}$$

the element of coordinate (not physical) length being $d\sigma$ and λ^m being unit tangent to the path in 3-space. Thus a particle trajectory or an integral curve of a vector field constitutes a two-dimensional spacetime on which we can assign coordinates $\eta^1 = \sigma$, $\eta^4 = x^4 = cT$, with T the coordinate time which is also the proper time of the observer launching and subsequently detecting the interference of the particle beams. Latin indices run from 1 to 3 in the four-dimensional spacetime. In the two-dimensional spacetime

$$
\begin{aligned}
&\mathrm{d}s^2 = G_{\alpha\beta}\,\mathrm{d}\eta^\alpha\,\mathrm{d}\eta^\beta \\
&G_{11} = 1 \qquad G_{14} = g_{m4}\lambda^m \qquad G_{44} = g_{44} \\
&G^{11} = Gg_{44} \qquad G^{14} = -Gg_{m4}\lambda^m \qquad G^{44} = G \equiv -[|g_{44}| + (g_{m4}\lambda^m)^2]^{-1}.
\end{aligned} \tag{3}
$$

In what follows it will be clear from the context whether we are working in this two-dimensional spacetime in which Greek indices take only the values 1 and 4 or in the four-dimensional spacetime.

The eikonal equation {(38) in Scorgie 1993} for the quantum mechanical wave can be written

$$\Lambda_\alpha \Lambda^\alpha = -(mc/\hbar)^2 \tag{4}$$

the rest mass of the particle being m and the wavevector for a wave having phase φ being

$$\Lambda^\alpha = G^{\alpha\beta}\varphi_{,\beta} \tag{5}$$

the comma denoting partial differentiation. The properties associated with the wavevector are expressed by writing it in the form

$$\Lambda^\alpha = q_1\zeta^\alpha + (q_2/c)\theta^\alpha \tag{6}$$

θ^α being the 4-velocity of the fixed coordinate point (η^1 fixed in the two-dimensional spacetime), ζ^α being unit 4-vector orthogonal to the 4-velocity, and q_1 and q_2 being reciprocal lengths which according to (4) are related by

$$q_1^2 - q_2^2 = -(mc/\hbar)^2. \tag{7}$$

In the two-dimensional coordinates

$$\zeta_\alpha = (G^{11})^{-1/2}\delta^1_\alpha \qquad \theta^\alpha = c\,|G_{44}|^{-1/2}\delta^\alpha_4. \tag{8}$$

To find the group velocity of the waves, using coordinate measures of distance σ and time T, we have the element of phase

$$\mathrm{d}\varphi = \Lambda_\alpha\,\mathrm{d}\eta^\alpha = \Lambda_1\,\mathrm{d}\sigma + \Lambda_4 c\,\mathrm{d}T.$$

Hence from (6) and (8) the wavenumber q and angular frequency ω are given in terms of the covariant components of the wavevector by

$$
\begin{aligned}
&q = \Lambda_1 = q_1(G^{11})^{-1/2} + q_2\,|G_{44}|^{-1/2}G_{14} \\
&\omega = -c\Lambda_4 = cq_2\,|G_{44}|^{1/2}.
\end{aligned} \tag{9}
$$

Expressing q_1 and q_2 in terms of q and ω, we can then regard (7) as the dispersion relation and so find the reciprocal of the group velocity to be

$$\mu_g = |cg_{44}|^{-1}\{g_{m4}\lambda^m + (q_2/q_1)[|g_{44}| + (g_{m4}\lambda^m)^2]^{1/2}\} \tag{10}$$

which is to be compared with the reciprocal velocity of the particle

$$\mu = |cg_{44}|^{-1}\{g_{m4}\lambda^m \pm n[|g_{44}| + (g_{m4}\lambda^m)^2]^{1/2}\} \tag{11}$$

which formed the basis for the treatment in the paper, the refractive index for the particle being n.

To show that (10) and (11) are identical for a free particle notice that unit tangent to its geodesic is $(\hbar/mc)\Lambda^\alpha$. The time component of the cotangent is a constant denoted by K^{-1} in the four-dimensional coordinates; and a covector has the same time component whether expressed in four-dimensional or two-dimensional coordinates. Hence we can write

$$K^{-1}=(\hbar/mc)\Lambda_4=-(\hbar/mc)q_2\,|g_{44}|^{1/2}. \tag{12}$$

Also it was shown in Scorgie (1993) that the refractive index for the free particle is

$$n=[1-K^2|g_{44}|]^{-1/2}. \tag{13}$$

Substituting from (12) gives

$$n=[1-(mc/\hbar q_2)^2]^{-1/2}. \tag{14}$$

Then (7) gives

$$q_2/q_1=\pm n \tag{15}$$

showing that (10) and (11) are identical.

The element of phase of the wave, $\Lambda_\alpha\,\mathrm{d}\eta^\alpha=\mathrm{d}\varphi$, is

$$\mathrm{d}\varphi=\omega\,|cg_{44}|^{-1}\{g_{m4}\lambda^m+(q_1/q_2)[|g_{44}|+(g_{m4}\lambda^m)^2]^{1/2}\}\,\mathrm{d}\sigma-\omega\,\mathrm{d}T. \tag{16}$$

From (9) and (12) the angular frequency

$$\omega=cq_2|g_{44}|^{1/2} \tag{17}$$

is constant in the two-dimensional spacetime.

To see how (16) works out in practice, consider the simple case where rotation of the observer's space axes is ignored ($g_{m4}=0$) and the interference results from two waves that travel along their respective paths having the same coordinate length σ but differing in gravitational potential by an amount $\Delta\psi$.

Equation (22) of Scorgie (1993) gave $n\sigma\omega\Delta\psi/c^3$ for the phase difference that causes interference. In the wave picture, from (16) and (17) and with

$$\left.\begin{aligned}&\mathrm{d}T=0,\\&\mathrm{d}\varphi=(\omega/c)(q_1/q_2)|g_{44}|^{-1/2}\,\mathrm{d}\sigma=q_1\,\mathrm{d}\sigma.\end{aligned}\right\} \tag{18}$$

Hence the difference between the phase increments over the two paths is

$$\Delta\varphi=\Delta q_1\sigma.$$

From (7) and (17) with $|g_{44}|^{1/2}=1-\psi/c^2$

$$\Delta\varphi=(q_2/q_1)\Delta q_2\sigma=n\sigma\omega\Delta\psi/c^3 \tag{19}$$

which is the value obtained in Scorgie (1993).

Of course in the wave picture the phase differences that produce interference are evaluated at constant coordinate time T as in this calculation. There is no question of finding the relevant phase increment by integrating (16) following the motion of the particle. Nevertheless it is interesting to ask what would be the result of such a calculation. To answer this question recall that the reciprocal of the particle velocity is given by (10). Hence to follow the motion of the particle we put $\mathrm{d}T=\mu_g\,\mathrm{d}\sigma$ in (16), obtaining

$$\mathrm{d}\varphi=-q_1(n^2-1)[1+|g_{44}|^{-1}(g_{m4}\lambda^m)^2]^{1/2}\,\mathrm{d}\sigma. \tag{20}$$

Now equation (4) of Scorgie (1993) gives the particle proper time element

$$d\tau = c^{-1}(n^2-1)^{1/2}[1+|g_{44}|^{-1}(g_{m4}\lambda^m)^2]^{1/2}\, d\sigma. \tag{21}$$

Hence following the motion of the particle

$$d\varphi = -(q_2^2-q_1^2)^{1/2}c\, d\tau = -(mc^2/\hbar)\, d\tau. \tag{22}$$

Apart from the negative sign this reproduces the identification $\varphi=(mc^2/\hbar)\tau$ mentioned in Scorgie (1993) where, in fact, the argument strictly allowed either sign. The negative sign is required if $-mc^2\tau$, regarded as the action, is to be a minimum on a free path, as pointed out by Landau and Lifshitz (1962).

These remarks, and the expression (16) for the element of phase, resolve a paradox that was mentioned in Scorgie (1993). The paradox comes about by applying (21) to a closed ring interferometer in which two beams of particles are launched from a common point fixed on the ring and travel in opposite directions round it to interfere at the launch point. Provided the refractive index (if not constant) is a function only of position on the ring integration of (21) shows that the proper time increment is the same for both directions of travel. Hence if proper time of the particle is accepted as a measure of the phase of the wave it would seem that the waves associated with the oppositely travelling particles must interfere constructively and independently of gravity and non-inertial motion of the ring. On the contrary, of course, the closed ring interferometer does respond to both of these influences. For example the first term in curly brackets in (16) gives rise to the Sagnac effect. Thus although particle proper time is a measure of the phase of the wave in the context of the argument leading to (22) it does not follow that equality of proper times over two paths implies equality of phase increments for assessing degree of interference of the associated waves.

Finally some general remarks may be made. Equation (16) provides an exact solution of the eikonal equation (4) in a form that is believed to be novel. An expression that in some respects is less explicit has been given by Anandan (1977) in terms of a time-like Killing vector as an auxiliary feature.

Although the two approaches discussed in this addendum (that based on representative particle motion and that based on the wave picture) are fully equivalent for free particles, there remains an interesting difference between them. The relations obtained for the wave picture assume free particles whereas the basic relation (11) for particle motion holds without restriction. Of course it is only for free particles that the refractive index in (11) has the value given by (14). This emphasizes the fact that (11) is a purely kinematic relation: the dynamics enters in determining the refractive index to be used in (11). These considerations suggest that the particle picture may plausibly be used in calculating the effect, on interference patterns, of departure from free particle motion, if the effect of the departure can be expressed through a change in the refractive index for the classical particle.

References

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